Limits

Definition: $\lim_{x \to a} f(x) = L$ means the value given by $f(x)$ approaches some real number L, as x approaches some value a. If a limit does not have a finite value, then we say that the limit does not exist.

What is the limit?	**Why?**
$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$	As x becomes very large, the function $f(x) = \frac{1}{x}$ becomes very small and approaches (but never actually equals) zero.
$\lim_{x \to 0} (\ln x)$ does not exist	As x gets closer and closer to 0, the $\ln x$ function approaches negative infinity without bound.

Right hand limit: $\lim_{x \to a^+} f(x) = L$ means the limit of $f(x)$ is L, as x approaches a from the right.
Left hand limit: $\lim_{x \to a^-} f(x) = L$ means the limit of $f(x)$ is L, as x approaches a from the left.

If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then the limit does not exist.

Continuity of a function at a point: A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$. This means that: ① $f(a)$ is defined; ② $\lim_{x \to a} f(x)$ exists; ③ $\lim_{x \to a} f(x) = f(a)$

Some Properties of Limits

If both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and k is a constant, then:

Constant Multiple Rule	$\lim_{x \to a}[kf(x)] = k\lim_{x \to a} f(x)$	$\lim_{x \to 4}[3x] = 3\lim_{x \to 4} x$
Sum/Difference Rule	$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$	$\lim_{x \to 3}\left[x^2 - \frac{2x}{x-2}\right] = \lim_{x \to 3} x^2 - \lim_{x \to 3} \frac{2x}{x-2}$
Product Rule	$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$	$\lim_{x \to 1}(x+3)(x-5) =$ $\lim_{x \to 1}(x+3) \cdot \lim_{x \to 1}(x-5)$
Quotient Rule	$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$	$\lim_{x \to 0}\left[\frac{3x-5}{e^x - 1}\right] = \frac{\lim_{x \to 0}(3x-5)}{\lim_{x \to 0}(e^x - 1)}$
Composition Rule	$\lim_{x \to a} f[g(x)] = f\left[\lim_{x \to a} g(x)\right]$	$\lim_{x \to 0^+} ln(2+3x) = ln\left[\lim_{x \to 0^+}(2+3x)\right]$ $\lim_{x \to 2}[5x^2 - 3]^{10} = \left[\lim_{x \to 2}(5x^2 - 3)\right]^{10}$ $\lim_{x \to 2}\left[\sqrt[3]{3^x - 9}\right] = \sqrt[3]{\lim_{x \to 2}(3^x - 9)}$

Strategies for Solving Limits

Continuous Functions	If $f(x)$ is continuous at a then $\lim_{x \to a} f(x) = f(a)$
Factor and Cancel	$\lim_{x \to 2} \frac{x^2+4x-12}{x^2-2x} = \lim_{x \to 2} \frac{(x-2)(x+6)}{x(x-2)} = \lim_{x \to 2} \frac{x+6}{x} = \frac{8}{2}$ for $x \neq 2$
Rationalize Numerator/Denominator	$\lim_{x \to 9} \frac{3-\sqrt{x}}{x^2-81} = \lim_{x \to 9} \frac{3-\sqrt{x}}{x^2-81} * \frac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x \to 9} \frac{9-x}{(x^2-81)(3+\sqrt{x})} = \lim_{x \to 9} \frac{-1}{(x+9)(3+\sqrt{x})} = \frac{-1}{108}$ for $x \neq 9$
L' Hospital's Rule	If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$
Polynomials at Infinity	$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$ factor out the largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then compute the limit. $\lim_{x \to -\infty} \frac{3x^2 - 4}{5x - 2x^2} = \lim_{x \to -\infty} \frac{x^2(3 - \frac{4}{x^2})}{x^2(\frac{5}{x} - 2)} = \lim_{x \to -\infty} \frac{3 - \frac{4}{x^2}}{\frac{5}{x} - 2} = -\frac{3}{2}$
Combine Rational Expressions	$\lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{-h}{x(x+h)}\right)$ $= \lim_{h \to 0}\left(\frac{-1}{x(x+h)}\right) = -\frac{1}{x^2}$
Piecewise Functions	$\lim_{x \to -2} g(x)$ where $g(x) = \begin{cases} x^2 + 5 \text{ if } x < -2 \\ 1 - 3x \text{ if } x \geq -2 \end{cases}$ Compute two one-sided limits $\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x^2 + 5 = 9$ $\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} 1 - 3x = 7$ The one-sided limits are different, therefore $\lim_{x \to -2} g(x)$ does not exist. If each limit was equal, then the limit would have existed and been equal to the same value.