MATH 184 Practice Problems

Name:____

1. Solve the following differential equations:

   a. $yy'' = 2(y')^2 + y^2, \quad y(0) = 1, \; y'(0) = 0$

   b. $y'' + 3y' - 10y = 14e^{2x}$

c. $2xydx + (x^2 + 1)dy = 0$

d. $\frac{dy}{dx} + \frac{1}{2}(\tan(x))y = 2y^3 \sin(x)$

e. $(3x - 2y)\frac{dy}{dx} = 3y$

2. Let $A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Be sure to show all work and justify your claims.

a. Find any/all eigenvalues of A

b. Find any corresponding eigenvectors of A

c. Find an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix

d. Calculate the trace of $(2I - 3A)^2$

e. Determine the inverse of the matrix $S^{-1}AS$

3. Suppose that $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ is a linear transformation such that

$$T(x^2 - x - 3) = \begin{pmatrix} -2 & 1 \\ -4 & -1 \end{pmatrix}, \; T(2x+5) = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}, \; T(6) = \begin{pmatrix} 12 & 6 \\ 6 & 18 \end{pmatrix}$$

Show that the set $\beta = \{x^2 - x - 3, 2x + 5, 6\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

1. a) $y \cdot y'' = 2(y')^2 + y^2$, $y(0) = 1$, $y'(0) = 0$

$$y\left(\frac{d^2y}{dx^2}\right) = 2\left(\frac{dy}{dx}\right)^2 + y^2 \Longrightarrow \frac{d^2y}{dx^2} = \frac{2}{y}\left(\frac{dy}{dx}\right)^2 + y$$

$$\begin{cases} v = \frac{dy}{dx} \\ v\frac{dv}{dy} = 2v^2 + y^2 \end{cases} \Longrightarrow v\frac{dv}{dy} - \frac{2}{y}v^2 = y$$

$$\frac{dv}{dy} - \frac{2}{y}v = yv^{-1}$$

Bernoulli: $\frac{du}{dy} + (1-n)p(y)u = (1-n)\,q(y)$

let $u = v^{1-(-1)} = v^2 \rightarrow v = u^{1/2}$

and $\frac{du}{dy} = 2v \cdot \frac{dv}{dy} \rightarrow \frac{dv}{dy} = \frac{1}{2v} \cdot \frac{du}{dy} = \frac{1}{2u^{1/2}} \cdot \frac{du}{dy}$

so: $\frac{dv}{dy} - \frac{2}{y}v = yv^{-1}$

$\hookrightarrow \frac{1}{2u^{1/2}} \cdot \frac{du}{dy} - \frac{2}{y}u^{1/2} = y(u^{1/2})^{-1} = yu^{-1/2}$

so: $\frac{du}{dy} - \frac{4}{y}u = 2y$ $\qquad$ <u>1^{st} order linear</u>

I.f. is $e^{-\int \frac{4}{y}dy} = e^{-4\ln|y|} = e^{\ln|y^{-4}|} = y^{-4}$

$$\frac{d}{dy}(y^{-4} \cdot u) = 2y \cdot y^{-4} = 2y^{-3}$$

$$y^{-4}u = \int 2y^{-3}dy = -y^{-2} + C_1$$

$$u = -y^2 + C_1y^4$$

now $v^2 = u \Rightarrow v = \pm\sqrt{u} = \pm\sqrt{C_1y^4 - y^2}$

so $\frac{dy}{dx} = \pm\sqrt{C_1y^4 - y^2}$

$$\boxed{\pm\int dx = \int \frac{1}{\sqrt{C_1y^4 - y^2}}dy}$$

1. b) $y'' + 3y' - 10y = 14e^{2x}$ ← Annihilator

$(D-2)$

2

$D^2 + 3D - 10$

$(D-2)(D+5)$

$2 \quad -5$

$c_1 e^{2x} + c_2 e^{-5x} + Axe^{2x}$

$y_p = Axe^{2x}$

$y'_p = Ae^{2x} + 2Axe^{2x}$

$y''_p = 2Ae^{2x} + 4Axe^{2x} + 2Ae^{2x} \Rightarrow 4Ae^{2x} + 4Axe^{2x}$

$(4Ae^{2x} + 4Axe^{2x}) + 3(Ae^{2x} + 2Axe^{2x}) - 10(Axe^{2x}) = 14e^{2x}$

$4Ae^{2x} + 4Axe^{2x} + 3Ae^{2x} + 6Axe^{2x} - 10Axe^{2x} = 14e^{2x}$

$7Ae^{2x} = 14e^{2x}$

$A = 2$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-5x} + 2xe^{2x}}$$

c) $\overbrace{2xy}^{M}dx + \overbrace{(x^2+1)}^{N}dy = 0$ $\qquad M_y = 2x \overset{\text{Exact}}{=} N_x = 2x$

$\int 2xy\,dx = x^2y + c_y$

$\int (x^2+1)\,dy = x^2y + y + c_x$

$$\boxed{f(x,y) = x^2y + y + c}$$

1. d) $\frac{dy}{dx} + \frac{1}{2}\tan(x)\, y = 2\sin(x) y^3$

Bernoulli w/ $n=3$

$p(x) = \frac{1}{2}\tan(x)$

$q(x) = 2\sin(x)$

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$$

Let $u = y^{1-n} \Rightarrow y^{1-3} = y^{-2}$

$u = \frac{1}{y^2}$

$$\frac{du}{dx} - 2\left(\tfrac{1}{2}\tan(x)\right)u = -2(2\sin(x))$$

$$\frac{du}{dx} - \tan(x)\,u = -4\sin(x)$$

$$I(x) = e^{\int -\tan(x)\,dx} = e^{\int -\frac{\sin x}{\cos x}dx} = e^{\ln(\cos(x))} = \cos(x)$$

$$(\cos(x)\cdot u)' = \cos(x)\cdot(-u\sin(x))$$

$$u = \frac{1}{\cos(x)}\int -4\cos(x)\sin(x)\,dx$$

$u = \sin(x)$

$du = \cos(x)\,dx$

$$u = \frac{1}{\cos(x)}\int -4u\,du$$

$$u = \frac{1}{\cos(x)}(-2u^2 + c) \Rightarrow u = \frac{1}{\cos(x)}(-2\sin^2(x) + c)$$

back sub u; $u = \frac{1}{y^2}$

$$\frac{1}{y^2} = \frac{1}{\cos(x)}(-2\sin^2(x) + c)$$

$$\boxed{\sqrt{\frac{\cos(x)}{-2\sin^2(x) + C}} = y}$$

1. e) $(3x-2y)\frac{dy}{dx}=3y$

$$x\left(3-2\frac{y}{x}\right)\frac{dy}{dx}=3y$$

$$\left(3-2\frac{y}{x}\right)\frac{dy}{dx}=3\frac{y}{x} \qquad v=\frac{y}{x} \qquad \begin{aligned} y&=vx \\ dy&=v+x\frac{dv}{dx}\end{aligned}$$

$$(3-2v)\left(v+x\frac{dv}{dx}\right)=3v$$

$$x\frac{dv}{dx}=\frac{3v}{3-2v}-v$$

$$x\frac{dv}{dx}=\frac{2v^2}{3-2v}$$

$$\int\frac{3-2v}{2v^2}dv=\int\frac{1}{x}dx$$

$$\int\left(\frac{3}{2v^2}-\frac{1}{v}\right)dv=\ln(x)+c$$

$$-\frac{3}{2}v^{-1}-\ln|v|=\ln|x|+c$$

$$-\frac{3}{2}v^{-1}-c=\ln|x|+\ln|v|$$

$$-\frac{3}{2}v^{-1}-c=\ln|x\cdot v| \qquad \begin{aligned} y&=vx \\ v&=\frac{y}{x}\end{aligned}$$

$$-\frac{3}{2}\left(\frac{y}{x}\right)^{-1}-c=\ln|y|$$

$$-\frac{3x}{2y}-c=\ln|y|$$

$$e^{-\frac{3x}{2y}-c}=y \qquad \boxed{ce^{-\frac{3x}{2y}}=y}$$

2. a) $\begin{vmatrix} \lambda-1 & 2 & 0 \\ 2 & \lambda-1 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} = 0$

$(\lambda-1)\begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 0 & \lambda-3 \end{vmatrix} = 0$

$(\lambda-1)^2(\lambda-3) - 4(\lambda-3) = 0$

$(\lambda-3)\left[(\lambda-1)^2-4\right] = 0$ $\qquad (\lambda-1)^2-4=0$

$\downarrow$ $\qquad (\lambda-1)^2=4$

$\lambda=3$ $\qquad \lambda-1=\pm 2 \rightarrow \lambda=-1, 3$

$\boxed{\lambda=-1 \quad \lambda=3 \text{ (mult. 2)}}$

b) $\lambda=-1$ $\quad \begin{bmatrix} (-1)-1 & 2 & 0 \\ 2 & (-1)-1 & 0 \\ 0 & 0 & (-1)-3 \end{bmatrix}$

$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow[\frac{R_3}{-4} \to R_3^*]{R_1+R_2 \to R_2^* \;\; \frac{R_1}{-2} \to R_1^*} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad x_1 - x_2 = 0 \rightarrow x_1 = x_2$

$x_2 = x_2$

$x_3 = 0$

$s(1,1,0) \rightarrow \boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

$\lambda=3$ $\quad \begin{bmatrix} 3-1 & 2 & 0 \\ 2 & 3-1 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2-R_1=R_2^*]{\frac{R_1}{2}=R_1^*} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1=-x_2$

$x_1+x_2=0$

$x_2=x_2$

$x_3=x_3$

$(-x_2, x_2, x_3) \rightarrow (s, s, t)$

$s(1,1,0) + t(0,0,1)$

$\boxed{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

2. c) The bases for the eigenspaces can be combined into a matrix that diagonalizes A

$$\boxed{S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\lambda = -1$ $\quad$ $\lambda = 3$

d) $2I - 3A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 0 \\ 6 & -1 & 0 \\ 0 & 0 & -7 \end{bmatrix}$

$(2I - 3A)^2 = \boxed{\begin{bmatrix} 37 & -12 & 0 \\ -12 & 37 & 0 \\ 0 & 0 & 49 \end{bmatrix}}$

e) $S^{-1}AS$ will give a diagonal matrix depending on the order you put the eigenvectors in

$S^{-1}AS = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$[S^{-1}AS]^{-1}$ the inverse of a diagonal matrix is the reciprocal of the diagonal

$[S^{-1}AS]^{-1} = \boxed{\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}$

3. a) $P_2(\mathbb{R}) = ax^2 + bx + c$

A set is a Basis iff it is linearly independent: LI if $\det(A) \neq 0$

$$ax^2 + bx + c = c_1(x^2 - x - 3) + c_2(2x+5) + c_3(6)$$

$$ax^2 + bx + c = c_1x^2 - c_1x - c_1 3 + c_2 2x + c_2 5 + c_3 6$$

$$\begin{cases} c_1x^2 & = ax^2 \\ -c_1x + c_2 2x & = bx \\ -c_1 3 + c_2 5 + c_3 6 & = c \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 5 & 6 \end{pmatrix}$$

column 3

$$\det\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 5 & 6 \end{pmatrix} \rightarrow 0\begin{pmatrix} -1 & 2 \\ -1 & 5 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

$$\rightarrow 6[(2)(1) - \cancel{(-1)(0)}^{0}]$$

$$\det\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 5 & 6 \end{pmatrix} = 6(2) = 12 \neq 0 \therefore$$ set β is a basis for $P_2(\mathbb{R})$

$\boxed{\text{The } \det\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 5 & 6 \end{pmatrix} \neq 0 \therefore \text{ set } \beta \text{ is a basis for } P_2(\mathbb{R})}$